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A study of intersections of quadrics having applications on the small weight codewords of the functional codes $C_2(Q)$, Q a non-singular quadric

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ABSTRACT

We study the small weight codewords of the functional code $C_2(Q)$, with Q a non-singular quadric in $PG(N, q)$. We prove that the small weight codewords correspond to the intersections of Q with the singular quadrics of $PG(N, q)$ consisting of two hyperplanes. We also calculate the number of codewords having these small weights.

1. Introduction

Consider a non-singular quadric Q of $PG(N, q)$. We denote the point set of Q by $Q = \{P_1, \ldots, P_n\}$, where we normalize the coordinates of the points P_i with respect to the leftmost non-zero coordinate. Let $\mathcal{F}$ be the set of all homogeneous quadratic polynomials $f(X_0, \ldots, X_N)$ defined by $N + 1$ variables.

The functional codes $C_2(Q)$ that are investigated in this article are inspired on the article of Lachaud [12] on linear codes defined on algebraic varieties. In general, for a fixed algebraic variety X in $PG(N, q)$, the functional code $C_h(X)$ is equal to

$$C_h(X) = \{(f(P_1), \ldots, f(P_n)) \| f \in \mathcal{F}_h\} \cup \{0\},$$

with $\mathcal{F}_h$ the set of the homogeneous polynomials of degree h over the finite field $\mathbb{F}_q$ in the variables $X_0, \ldots, X_N$.

So, in particular, the functional code $C_2(Q)$ is the linear code

$$C_2(Q) = \{(f(P_1), \ldots, f(P_n)) \| f \in \mathcal{F} \cup \{0\}\},$$

defined over $\mathbb{F}_q$.

This linear code has length $n = |Q|$ and dimension $k = \binom{N+2}{2} - 1$. The third fundamental parameter of this linear code $C_2(Q)$ is its minimum distance d.

We determine the five or six smallest weights of $C_2(Q)$ via geometrical arguments. Every homogeneous quadratic polynomial f in $N + 1$ variables defines a quadric $Q' : f(X_0, \ldots, X_N) = 0$. The small weight codewords of $C_2(Q)$ correspond to the quadrics of $PG(N, q)$ having the largest intersections with Q.

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We prove that these small weight codewords correspond to quadrics Q' which are the union of two hyperplanes of $PG(N, q)$. Since there are different possibilities for the intersection of two hyperplanes with a non-singular quadric, we determine in this way the five or six smallest weights of the functional code $C_2(Q)$.

We also determine the exact number of codewords having the five or six smallest weights.

In this way, we continue the work of Lachaud [12] on linear codes defined on algebraic varieties, and the work of Edoukou [3,4]. In [3,4], Edoukou investigated the functional codes arising from the intersections of quadrics with the non-singular Hermitian variety in $PG(3, q^2)$ and $PG(4, q^2)$, and the functional codes arising from the intersections of quadrics with the non-singular quadrics and the quadratic cone in $PG(3, q)$. In [8], Hallez and Storme continued this study on the functional codes arising from the intersections of quadrics with the non-singular Hermitian variety in $PG(N, q^2)$, $N < O(q^2)$.

A similar investigation was performed in [5], where the small weight codewords of the q-ary functional code $C_{herm}(X)$, arising from the intersections of a non-singular Hermitian variety X in $PG(N, q^2)$ with the Hermitian varieties in $PG(N, q^2)$, were investigated.

Regarding the divisibility results of Theorems 4.3 and 4.4, we also wish to refer to the divisibility results mentioned in [6].

In the preceding paragraphs, we focused on similar results on small weight codewords in functional codes. We wish however to mention that the study of small weight codewords in different classes of linear codes related to geometrical structures has received great attention in the recent literature. We mention in particular the results of [2] on linear codes defined by conics in $PG(2, q)$, the results of [11,16–18] on small weight codewords related to generalized quadrangles and classical polar spaces, the results on the small weight codewords of the linear codes related to the incidence matrices of $PG(N, q)$ [7,13–15,22], and the results on the small weight codewords of the d-th-order projective Reed–Muller codes $PRM(q, d, n)$ [19–21].

Some of these results were motivated by the recent interest in LDPC (Low Density Parity-check Codes) codes [2,11,16–18]. In all of these cases, the problem of investigating the small weight codewords of these linear codes was translated into a geometrical problem in finite projective spaces, thereby illustrating the use of geometrical methods for obtaining new results on the small weight codewords in linear codes related to geometrical structures.

We also mention the theorem of Bézout which will be used frequently.

Theorem 1.1. *Let X and Y be algebraic subvarieties in $PG(N, L)$, L an algebraically closed field, of pure dimension k and l with $k + l \geqslant N$, and suppose that they intersect generically transversely. Then $\deg(X \cap Y) = \deg(X) \cdot \deg(Y)$.*

In particular, if $k + l = N$, this means that $X \cap Y$ consists of $\deg(X) \cdot \deg(Y)$ points, where the different intersection points are counted according to their intersection multiplicities. A pair of pure-dimensional varieties X and Y contained in $PG(N, L)$ which intersect properly in their intersection has the expected dimension for their intersection, i.e., $\dim(X \cap Y) = \dim(X) + \dim(Y) - N$.

This theorem of Bézout reduces to the following theorem on the intersection of an algebraic variety in projective space with a hypersurface.

Theorem 1.2. *Let V be an algebraic variety of dimension at least 1 in the projective space $PG(N, L)$ of dimension N over the algebraically closed field L, and let H be a hypersurface in $PG(N, L)$ not containing V.*

Let $X_1, \ldots, X_s$ be the irreducible components of $V \cap H$; then

$$\sum_{i=1}^{s} i(V, H; X_j) \cdot \deg X_j = \deg V \cdot \deg H,$$

with $i(V, H; X_j)$ the intersection multiplicity of the varieties V and H in the component X_j.

We will apply this theorem of Bézout in the following context.

Corollary 1.3. *If a quadratic variety V of dimension at least 1 intersects a quadratic hypersurface H in $PG(N, q)$ in more than four irreducible components $X_1, \ldots, X_s$, then this quadratic variety V is completely contained in the quadratic hypersurface H.*

2. Quadrics in $PG(N, q)$

The non-singular quadrics in $PG(N, q)$ are equal to:

- The non-singular parabolic quadrics $Q(2N, q)$ in $PG(2N, q)$ having standard equation $X_0^2 + X_1X_2 + \cdots + X_{2N-1}X_{2N} = 0$. These quadrics contain $q^{2N-1} + \cdots + q + 1$ points, and the largest dimensional spaces contained in a non-singular parabolic quadric of $PG(2N, q)$ have dimension $N - 1$.
- The non-singular hyperbolic quadrics $Q^+(2N+1, q)$ in $PG(2N+1, q)$ having standard equation $X_0X_1 + \cdots + X_{2N}X_{2N+1} = 0$. These quadrics contain $(q^N + 1)(q^{N+1} - 1)/(q - 1) = q^{2N} + q^{2N-1} + \cdots + q^{N+1} + 2q^N + q^{N-1} + \cdots + q + 1$ points, and the largest dimensional spaces contained in a non-singular hyperbolic quadric of $PG(2N + 1, q)$ have dimension N.
- The non-singular elliptic quadrics $Q^-(2N + 1, q)$ in $PG(2N + 1, q)$ having standard equation $f(X_0, X_1) + X_2X_3 + \cdots + X_{2N}X_{2N+1} = 0$, where $f(X_0, X_1)$ is an irreducible quadratic polynomial over $\mathbb{F}_q$. These quadrics contain $(q^{N+1} + 1)(q^N - 1)/(q - 1) = q^{2N} + q^{2N-1} + \cdots + q^{N+1} + q^{N-1} + \cdots + q + 1$ points, and the largest dimensional spaces contained in a non-singular elliptic quadric of $PG(2N + 1, q)$ have dimension $N - 1$.

Each of the quadrics of $\mathrm{PG}(N, q)$, including the non-singular quadrics, can be described as a quadric having an s-dimensional vertex π_s of singular points, $s \geqslant -1$, and having a non-singular base Q_{N-s-1} in an $(N-s-1)$-dimensional space skew to π_s, denoted by $\pi_s\mathrm{Q}_{N-s-1}$.

We denote the largest dimensional spaces contained in a quadric as the *generators* of this quadric.

Since we will make heavy use of the sizes of (non-)singular quadrics of $\mathrm{PG}(N, q)$, we list these sizes explicitly.

- In $\mathrm{PG}(N, q)$, a quadric having an $(N-2d-2)$-dimensional vertex and a hyperbolic quadric $\mathrm{Q}^+(2d+1, q)$ as base has size
$$q^{N-1} + \cdots + q^{N-d} + 2q^{N-d-1} + q^{N-d-2} + \cdots + q + 1.$$
- In $\mathrm{PG}(N, q)$, a quadric having an $(N-2d-2)$-dimensional vertex and an elliptic quadric $\mathrm{Q}^-(2d+1, q)$ as base has size
$$q^{N-1} + \cdots + q^{N-d} + q^{N-d-2} + \cdots + q + 1.$$
- In $\mathrm{PG}(N, q)$, a quadric having an $(N-2d-1)$-dimensional vertex and a parabolic quadric $\mathrm{Q}(2d, q)$ as base has size
$$q^{N-1} + q^{N-2} + \cdots + q + 1.$$

We note that the size of a (non-)singular quadric having a non-singular hyperbolic quadric as base is always larger than the size of a (non-)singular quadric having a non-singular parabolic quadric as base, which is itself always larger than the size of a (non-)singular quadric having a non-singular elliptic quadric as base.

The quadrics having the largest size are the unions of two distinct hyperplanes of $\mathrm{PG}(N, q)$, and have size $2q^{N-1} + q^{N-2} + \cdots + q + 1$. The second-largest quadrics in $\mathrm{PG}(N, q)$ are the quadrics having an $(N-4)$-dimensional vertex and a non-singular three-dimensional hyperbolic quadric $\mathrm{Q}^+(3, q)$ as base. These quadrics have size $q^{N-1} + 2q^{N-2} + q^{N-3} + \cdots + q + 1$. The third-largest quadrics in $\mathrm{PG}(N, q)$ have an $(N-6)$-dimensional vertex and a non-singular hyperbolic quadric $\mathrm{Q}^+(5, q)$ as base. These quadrics have size $q^{N-1} + q^{N-2} + 2q^{N-3} + q^{N-4} + \cdots + q + 1$.

As we mentioned in the introduction, the smallest weight codewords of the code $C_2(\mathrm{Q})$ correspond to the largest intersections of Q with other quadrics Q′ of $\mathrm{PG}(N, q)$. Let V be the intersection of the quadric Q with the quadric Q′. Two distinct quadrics Q and Q′ define a unique pencil of quadrics $\lambda\mathrm{Q} + \mu\mathrm{Q}'$, $(\lambda, \mu) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}$.

Let $V = \mathrm{Q} \cap \mathrm{Q}'$; then V also lies in every quadric $\lambda\mathrm{Q} + \mu\mathrm{Q}'$ of the pencil of quadrics defined by Q and Q′. A large intersection implies that there is a large quadric in the pencil. The sum of the numbers of points in the $q+1$ quadrics of the pencil of quadrics defined by Q and Q′ is $|\mathrm{PG}(N, q)| + q|V|$ points, since the points of V lie in all the quadrics of the pencil and the other points of $\mathrm{PG}(N, q)$ lie in exactly one such quadric. So there is a quadric in the pencil containing at least $(|\mathrm{PG}(N, q)| + q|V|)/(q+1)$ points.

If there is a quadric in the pencil which is equal to the union of two hyperplanes, then we are at the desired conclusion that the largest intersections of Q arise from the intersections of Q with the quadrics which are the unions of two hyperplanes. So assume that all $q+1$ quadrics in this pencil defined by Q and Q′ are irreducible; we try to find a contradiction. As already mentioned above, the largest irreducible quadrics are cones with vertex $\mathrm{PG}(N-4, q)$ and base $\mathrm{Q}^+(3, q)$, and the second-largest irreducible quadrics are cones with vertex $\mathrm{PG}(N-6, q)$ and base $\mathrm{Q}^+(5, q)$.

Theorem 2.1. *In $\mathrm{PG}(N, q)$, with $N \geqslant 6$, or $N = 5$ and $\mathrm{Q} = \mathrm{Q}^-(5, q)$, if $|V| > q^{N-2} + 3q^{N-3} + 3q^{N-4} + 2q^{N-5} + \cdots + 2q + 1$, then in the pencil of quadrics defined by Q and Q′, there is a quadric consisting of two hyperplanes.*

Proof. Suppose that there is no quadric consisting of two hyperplanes in the pencil of quadrics.

If $|V| > q^{N-2} + 2q^{N-3} + 2q^{N-4} + q^{N-5} + \cdots + q + 1$, then $(|\mathrm{PG}(N, q)| + q|V|)/(q+1) > |\pi_{N-6}\mathrm{Q}^+(5, q)|$, so there is a singular quadric $\pi_{N-4}\mathrm{Q}^+(3, q)$ in the pencil of quadrics.

With the lines of one regulus of $\mathrm{Q}^+(3, q)$, together with π_{N-4}, we form $q+1$ different $(N-2)$-spaces π_{N-2}. We wish to have that at least one of these $(N-2)$-spaces intersects Q in two $(N-3)$-dimensional spaces. All points of V appear in at least one of these $(N-2)$-dimensional spaces Π_{N-2}, so for some space π_{N-2}, we have that $|\pi_{N-2} \cap V| \geqslant |V|/(q+1)$.

If $|V|/(q+1) > |\pi_{N-6}\mathrm{Q}^+(3, q)|$, then $\pi_{N-2} \cap \mathrm{Q}$ is the union of two $(N-3)$-spaces. When $|V| > q^{N-2} + 3q^{N-3} + 3q^{N-4} + 2q^{N-5} + \cdots + 2q + 1$, then this is valid. So $\pi_{N-2} \cap \mathrm{Q} = \pi^1_{N-3} \cup \pi^2_{N-3}$.

These two $(N-3)$-dimensional spaces are contained in V, and so belong to Q. This means that Q must have subspaces of dimension $N-3$. The next table shows that this can only occur in small dimensions.

Quadric	Dimension generator	Property fulfilled
$\mathrm{Q} = \mathrm{Q}^+(N = 2N'+1, q)$	N'	$N' \leqslant 2$
$\mathrm{Q} = \mathrm{Q}^-(N = 2N'+1, q)$	$N'-1$	$N' \leqslant 1$
$\mathrm{Q} = \mathrm{Q}(N = 2N', q)$	$N'-1$	$N' \leqslant 2$

Except for the small cases for N', we have a contradiction, so there is a quadric consisting of two hyperplanes in the pencil of quadrics defined by Q and Q′. □

Remark 2.2. First of all we say something about the sharpness of the bound in Theorem 2.1. Therefore we refer the reader to [1, Theorem 3.6]. In a pencil of $q+1$ non-singular elliptic quadrics $\mathrm{Q}^-(N,q)$ (not containing hyperplanes), the size of the intersection of two quadrics is

$$|\mathrm{Q}_1 \cap \mathrm{Q}_2| = q^{N-2} + q^{N-3} + \cdots + q^{\frac{N+1}{2}} + q^{\frac{N-5}{2}} + \cdots + q + 1.$$

We notice that the difference between the size of this intersection and the bound mentioned in Theorem 2.1 is of order $O(q^{N-3})$.

Since the problem is solved for dimensions N up to 4 [3,4], there is only one case still open. From now on, Q will be the hyperbolic quadric $\mathrm{Q}^+(5,q)$.

If $|V| > q^3 + 2q^2 + 2q + 1$, then there is a singular quadric $\pi_{N-4}\mathrm{Q}^+(3,q) = L\mathrm{Q}^+(3,q)$ in the pencil of quadrics if we assume that there is no quadric in the pencil which is the union of two hyperplanes.

We form solids $\omega_1, \ldots, \omega_{q+1}$ with L and the lines of one regulus of the base $\mathrm{Q}^+(3,q)$. If $|V| > q^3 + 3q^2 + 3q + 1$, $|V|/(q+1) > |\pi_{N-6}\mathrm{Q}^+(3,q)|$, so there is a solid through L of $L\mathrm{Q}^+(3,q)$ intersecting Q in two planes.

Now we have three different cases:

1. $L \subset V$,
2. $|L \cap V| = 1$,
3. $|L \cap V| = 2$.

Lemma 2.3. *For $\mathrm{Q}^+(5,q)$, if $|V| > q^3 + 4q^2 + 1$ and $L \subset V$, then there is a quadric consisting of two hyperplanes in the pencil of quadrics defined by Q and Q′.*

Proof. Assume that no quadric in the pencil is the union of two hyperplanes. Then we have already a singular quadric $L\mathrm{Q}^+(3,q)$ in the pencil and there is a solid ω_1 through L intersecting Q in two planes. Now L lies in one or both of these planes, since $L \subset V$.

Every point of V lies in at least one of the $q+1$ solids $\omega_1, \ldots, \omega_{q+1}$ through L. Now

$$|V| - (\text{union of two planes}) > q^3 + 4q^2 + 1 - (2q^2 + q + 1) = q^3 + 2q^2 - q.$$

So one of the q remaining solids of $\omega_2, \ldots, \omega_{q+1}$ contains at least

$$|L| + \frac{q^3 + 2q^2 - q}{q} = q^2 + 3q$$

points.

So one solid ω_2 contains more than $|\mathrm{Q}^+(3,q)|$ points of V, so ω_2 intersects Q in the union of two planes. One of these planes contains L, so L lies already in two planes of $\mathrm{Q}^+(5,q)$.

Now one of the $q-1$ remaining solids $\omega_3, \ldots, \omega_{q+1}$ contains more than

$$q + 1 + (q^3 + 2q^2 - q - 2q^2)/(q-1) = q^2 + 2q + 1$$

points of V.

Again this implies that there is a solid ω_3 intersecting Q in the union of two planes, with at least one of them containing L. This gives us at least three planes of $\mathrm{Q}^+(5,q)$ through L, which is impossible. We have a contradiction. So there is a quadric consisting of two hyperplanes in the pencil of quadrics defined by Q and Q′. □

Lemma 2.4. *For $\mathrm{Q}^+(5,q)$, if $|V| > q^3 + 5q^2 + 1$, then the case $|L \cap V| = 1$ does not occur.*

Proof. Assume that no quadric in the pencil of Q and Q′ is the union of two hyperplanes. Then we have already a singular quadric $L\mathrm{Q}^+(3,q)$ in the pencil of quadrics. In this quadric, the line L is skew to the solid of $\mathrm{Q}^+(3,q)$.

But L is a line tangent to $\mathrm{Q}^+(5,q)$ at a point R since L is contained in the cone $L\mathrm{Q}^+(3,q)$, but L shares only one point with $\mathrm{Q}^+(5,q)$.

Using the same arguments as in the preceding lemma, we prove that at least three solids defined by the line L and lines of one regulus of the base $\mathrm{Q}^+(3,q)$ intersect Q in two planes. These planes all pass through R, so they lie in the tangent hyperplane $T_R(\mathrm{Q})$, which intersects Q in a cone with vertex R and base $\mathrm{Q}^+(3,q)'$. Two such planes of V in the same solid of $L\mathrm{Q}^+(3,q)$ through L intersect in a line, so they define lines of the opposite reguli of the base $\mathrm{Q}^+(3,q)'$ of this tangent cone. This shows that the 4-space defined by R and the base $\mathrm{Q}^+(3,q)'$ already shares six planes with Q. By Corollary 1.3, the cone $R\mathrm{Q}^+(3,q)'$ is contained in V.

Consider a hyperplane through L; this intersects $L\mathrm{Q}^+(3,q)$ either in a cone $L\mathrm{Q}(2,q)$ or in the union of two solids. So the tangent hyperplane $T_R(\mathrm{Q})$ cannot intersect $L\mathrm{Q}^+(3,q)$ in a cone $R\mathrm{Q}^+(3,q)'$.

This gives us a contradiction. □

Lemma 2.5. *For $\mathrm{Q}^+(5,q)$, if $|V| > q^3 + 5q^2 - q + 1$ and $|L \cap V| = 2$, then there is a quadric consisting of two hyperplanes in the pencil of quadrics defined by Q and Q′.*

Proof. Assume that no quadric in the pencil defined by Q and Q′ is the union of two hyperplanes. Then we have already a singular quadric $LQ^+(3,q)$ in the pencil and there is a solid ω_1 through L intersecting $Q = Q^+(5,q)$ in two planes. Assume that $L \cap V = \{R, R'\}$. The polarity of $Q^+(5,q)$ maps the bisecant L to its three-dimensional polar space π_3 with respect to $Q^+(5,q)$. This three-dimensional space π_3 intersects $Q^+(5,q)$ in a three-dimensional hyperbolic quadric $Q^+(3,q)_L$, called the polar quadric of L with respect to $Q^+(5,q)$. Then the cones with vertex R and R', respectively, and base $Q^+(3,q)_L$ are contained in $Q^+(5,q)$.

By the same counting arguments as in Lemma 2.3, we know that if $|V| > q^3 + 5q^2 - 2q + 2$, then there are three solids $\langle L, L_i\rangle$, with $i = 1, 2, 3$, and all L_i belonging to the same regulus of $Q^+(3,q)$, intersecting Q in two planes. For every solid $\langle L, L_i\rangle$, we denote by $\tilde{L_i}$ the line that the two planes have in common, and $\pi_{i1} = \langle R, \tilde{L_i}\rangle, \pi_{i2} = \langle R', \tilde{L_i}\rangle$. Then $\tilde{L_i} = \pi_{i1} \cap \pi_{i2} \subset R^\perp \cap R'^\perp = \pi_3$, with $\perp$ the polarity with respect to $Q^+(5,q)$. We use the same arguments for the opposite regulus. This gives us again three solids $\langle L, M_i\rangle$, $i = 1, 2, 3$, intersecting Q in two planes. We denote by $\tilde{M_i}$ the line in the intersection of these two planes.

These lines $\tilde{L_i}$ and $\tilde{M_i}$ belong to the hyperbolic quadric $Q^+(3,q)_L$ in $R^\perp \cap R'^\perp$, which is the basis for $RQ^+(3,q)_L$ as well as for $R'Q^+(3,q)_L$. The quadric $RQ^+(3,q)_L$ shares six planes with $LQ^+(3,q)$. By Theorem 1.2, if $RQ^+(3,q)_L \not\subset LQ^+(3,q)$, then the intersection would be of degree 4, so $RQ^+(3,q)_L \subset LQ^+(3,q) \cap Q$. Similarly, $R'Q^+(3,q)_L \subset LQ^+(3,q) \cap Q$.

The cone $LQ^+(3,q)$ intersects Q in two tangent cones $RQ^+(3,q)_L$ and $R'Q^+(3,q)_L$. We will now look at the pencil of quadrics defined by Q and $LQ^+(3,q) = Q'$.

Let P be a point of $\pi_3 \backslash Q^+(3,q)_L$. The points of $PG(5,q)\backslash(Q \cap Q')$ lie in exactly one quadric of the pencil defined by Q and Q′. For the point P, this must be the quadric consisting of the two hyperplanes $\langle R, \pi_3\rangle$ and $\langle R', \pi_3\rangle$, for $\langle R, \pi_3\rangle$ contains a cone $RQ^+(3,q)_L$ and the point P of this quadric, so this is one point too many for a quadric.

So one quadric of the pencil consists of the union of two hyperplanes. □

Corollary 2.6. *For $Q^+(5,q)$, if $|V| > q^3 + 5q^2 + 1$, then the intersection of $Q^+(5,q)$ with the other quadric Q′ is equal to the intersection of $Q^+(5,q)$ with the union of two hyperplanes.*

3. Dimension 4

We consider a pencil of quadrics $\lambda Q + \mu Q'$ in $PG(4,q)$, with Q a non-singular parabolic quadric $Q(4,q)$. Let $V = Q \cap Q'$. If $|V| > q^2 + q + 1$; then there is at least one cone $PQ^+(3,q)$ in this pencil.

Lemma 3.1. *If $|V| > q^2 + (x+1)q + 1$, then x planes through P of the same regulus of $PQ^+(3,q)$ intersect Q in two lines.*

Proof. Consider one regulus of $PQ^+(3,q)$. We wish to have that x planes PL, with L a line of this regulus, intersect Q in two lines. So for the first plane, this means that $\frac{|V|}{q+1} > q+1$, since every point of V lies in one of the $q+1$ planes PL. For the x-th plane, we have already $x-1$ planes which intersect Q in two lines. We impose that $\frac{|V|-(x-1)(2q+1)}{q-x+2} > q+1$ to guarantee that the x-th plane also intersects Q in two lines. This reduces to $|V| > q^2 + (x+1)q + 1$. □

Denote by L_i the lines of one regulus of $Q^+(3,q)$ and by M_i the lines of the opposite regulus of $Q^+(3,q)$, with $i = 1, 2, \ldots, q+1$. Denote by l_{i1}, l_{i2} (resp. m_{i1}, m_{i2}) the lines of $Q \cap PL_i$ (resp. $Q \cap PM_i$).

We have to look at two cases now, whether $P \in V$ or whether $P \notin V$.

CASE I: $P \in V$

Theorem 3.2. *For $Q(4,q)$, if $|V| > q^2 + 6q + 1$ and $P \in V$, then V consists of the union of a cone $PQ(2,q)$ and another three-dimensional quadric.*

Proof. If we consider one regulus of the base $PQ^+(3,q)$, then, by the preceding lemma, there are $x \geqslant 5$ planes each containing two lines of V, of which at least one goes through P. This gives us at least $x \geqslant 5$ lines through P in $Q(4,q) \cap PQ^+(3,q)$. These x lines lie on the tangent cone $PQ(2,q)$ in $T_P(Q(4,q))$. By Corollary 1.3, since $x \geqslant 5$, this cone $PQ(2,q)$ lies completely in $Q(4,q)$ and in $PQ^+(3,q)$.

Since $Q(4,q) \cap PQ^+(3,q)$ is an algebraic variety of degree 4 and dimension 2, and since $|V| > |PQ(2,q)|$, V is the union of $PQ(2,q)$ and another three-dimensional quadric. □

CASE II: $P \notin V$

Theorem 3.3. *For $Q(4,q)$, if $|V| > q^2 + 11q + 1$ and $P \notin V$, then for $q > 7$, V consists of the union of two hyperbolic quadrics.*

Proof. We use the notation introduced after the proof of Lemma 3.1.

Without loss of generality, we can assume that the lines of PL_i lying on Q intersected by m_{11} (resp. m_{12}) are the lines l_{i1} (resp. l_{i2}), $i = 1, \ldots, x$. So m_{11} and m_{12} are both intersected by x lines of Q.

The line m_{21} will intersect at least $\lceil \frac{x}{2} \rceil$ of the lines l_{i1}. This means that m_{21} has these transversals in common with m_{11}. Assume that these lines are the lines $l_{11}, \cdots, l_{\lceil \frac{x}{2} \rceil 1}$. Also m_{31} has at least $\lceil \frac{x}{2} \rceil$ transversals in common with m_{11}.

Assume that at least two of those transversals also intersect m_{21}; then m_{11}, m_{21}, m_{31} define a three-dimensional hyperbolic quadric $Q^+(3,q)$ sharing five lines with $Q(4,q)$.

Otherwise, at least $x-1$ transversals out of the x selected transversals to m_{11} are intersecting one of m_{21} and m_{31}. Suppose now that m_{41} shares at least $\lceil \frac{x}{2} \rceil$ transversals with m_{11}. One of them could be skew to m_{21} and m_{31}, but at least $\lceil \frac{x}{2} \rceil - 1$ of them intersect m_{21} or m_{31}. At least $\frac{\frac{x}{2}-1}{2}$ of them intersect, for instance, m_{21}. If this is at least 2, then m_{11}, m_{21}, m_{41} define a three-dimensional hyperbolic quadric $Q^+(3, q)$ sharing five lines with $Q(4, q)$. Therefore, we obtain the same conclusion that V contains a three-dimensional hyperbolic quadric when $x \geqslant 10$. Lemma 3.1 implies that we need to impose that $|V| > q^2 + 11q + 1$. Since in both cases, there is a three-dimensional hyperbolic quadric $Q^+(3, q)$ sharing five lines with $Q(4, q)$, Corollary 1.3 implies that $Q^+(3, q) \subset Q(4, q)$. So V consists of $Q^+(3, q)$ and another three-dimensional quadric. The remaining lines of V are ten skew lines of planes PL_i and ten skew lines of planes PM_j, and these lines of V lying in PL_i intersect the lines of V lying in PM_j. So these lines also form a three-dimensional hyperbolic quadric $Q^+(3, q)$. □

Theorem 3.4. *For $Q(4, q)$, if $|V| > q^2 + 11q + 1$, then there is a union of two hyperplanes in the pencil of quadrics defined by Q and Q'.*

Proof. By Theorems 3.2 and 3.3, V consists of a three-dimensional hyperbolic quadric $Q^+(3, q)$ in a solid π_3 and another three-dimensional quadric. Let R be a point of $\pi_3 \setminus V$. The points of $PG(4, q) \setminus (Q \cap Q')$ lie in exactly one quadric of the pencil. Let Q'' be the unique quadric in the pencil defined by Q and Q' containing R. So π_3 shares with Q'' a quadric and an extra point R, so this is one point too many for a quadric; hence there is a quadric in the pencil defined by Q and Q' containing a hyperplane, and so a quadric in the pencil defined by two hyperplanes. □

4. Tables

For the standard properties and notation for quadrics, we refer the reader to [9, Chapter 22].

We proved in Theorem 2.1, Corollary 2.6, and Theorem 3.4 that the small weight codewords of $C_2(Q)$, for Q a non-singular quadric in $PG(N, q)$, $N \geqslant 4$, correspond to the intersections of Q with the quadrics consisting of the union of two hyperplanes. We now count the number of codewords obtained via the intersections of Q with the union of two hyperplanes.

Consider a quadric Q' which is a union of two hyperplanes; then Q' defines $q-1$ codewords of $C_2(Q)$ equal to each other up to a non-zero scalar multiple.

Then Q and Q' define a pencil of quadrics in $PG(N, q)$, $N \geqslant 4$. A counting argument proves that for $N \geqslant 4$ and $q \geqslant 4$, this pencil cannot contain another quadric Q'' which is the union of two hyperplanes. So no other quadric Q'' which is the union of two hyperplanes leads to the same codewords of $C_2(Q)$ as the union Q' of two hyperplanes.

Hence, to calculate the number of codewords arising from the union of two hyperplanes, we simply check which unions of two hyperplanes determine codewords of a particular weight (Tables 1 and 2, Tables 4 and 5, Tables 7 and 8); we then count how many such pairs of hyperplanes there are in $PG(N, q)$, and then we multiply this number by $q-1$ since a union of two hyperplanes defines $q-1$ non-zero codewords which are scalar multiples of each other. For $N \geqslant 4$ and $q \geqslant 4$, this determines the precise number of codewords of the smallest weights in $C_2(Q)$ (Tables 3, 6, 9 and 10).

4.1. *The hyperbolic quadric in $PG(2l + 1, q)$*

We know that the largest intersections of a non-singular hyperbolic quadric $Q^+(2l+1, q)$ in $PG(2l+1, q)$ with the other quadrics are the intersections of $Q^+(2l+1, q)$ with the quadrics which are the union of two hyperplanes Π_1 and Π_2. We now discuss all the different possibilities for the intersections of $Q^+(2l+1, q)$ with the union of two hyperplanes. This then gives the five or six smallest weights of the functional codes $C_2(Q^+(2l+1, q))$, and the numbers of codewords having these weights.

We start the discussion via the $(2l-1)$-dimensional space $\Pi_{2l-1} = \Pi_1 \cap \Pi_2$. The intersection of a $(2l-1)$-dimensional space with the non-singular hyperbolic quadric $Q^+(2l+1, q)$ in $PG(2l+1, q)$ is: (1) a non-singular hyperbolic quadric $Q^+(2l-1, q)$, (2) a cone $LQ^+(2l-3, q)$, (3) a cone $PQ(2l-2, q)$, or (4) a non-singular elliptic quadric $Q^-(2l-1, q)$.

1. Let $PG(2l-1, q)$ be a $(2l-1)$-dimensional space intersecting $Q^+(2l+1, q)$ in a non-singular $(2l-1)$-dimensional hyperbolic quadric $Q^+(2l-1, q)$. Then $PG(2l-1, q)$ is the polar space of a bisecant line to $Q^+(2l+1, q)$. Then $PG(2l-1, q)$ lies in two hyperplanes tangent to $Q^+(2l+1, q)$ and in $q-1$ hyperplanes intersecting $Q^+(2l+1, q)$ in a non-singular parabolic quadric $Q(2l, q)$.
2. Let $PG(2l-1, q)$ be a $(2l-1)$-dimensional space intersecting $Q^+(2l+1, q)$ in a singular quadric $LQ^+(2l-3, q)$; then $PG(2l-1, q)$ lies in the hyperplanes tangent to $Q^+(2l+1, q)$ at the $q+1$ points P of L.
3. Let $PG(2l-1, q)$ be a $(2l-1)$-dimensional space intersecting $Q^+(2l+1, q)$ in a singular quadric $PQ(2l-2, q)$; then $PG(2l-1, q)$ lies in the hyperplane tangent to $Q^+(2l+1, q)$ in P, and in q hyperplanes intersecting $Q^+(2l+1, q)$ in non-singular parabolic quadrics $Q(2l, q)$.
4. Let $PG(2l-1, q)$ be a $(2l-1)$-dimensional space intersecting $Q^+(2l+1, q)$ in a non-singular $(2l-1)$-dimensional elliptic quadric $Q^-(2l-1, q)$; then $PG(2l-1, q)$ lies in $q+1$ hyperplanes intersecting $Q^+(2l+1, q)$ in non-singular parabolic quadrics $Q(2l, q)$.

Table 1

		$\Pi_{2l-1} \cap \mathrm{Q}^+(2l+1,q)$	$\lvert \mathrm{Q}^+(2l+1,q) \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$\mathrm{Q}^+(2l-1,q)$	$2\lvert \mathrm{Q}(2l,q)\rvert - \lvert \mathrm{Q}^+(2l-1,q)\rvert$
	(1.2)	$\mathrm{Q}^+(2l-1,q)$	$\lvert P\mathrm{Q}^+(2l-1,q)\rvert + \lvert \mathrm{Q}(2l,q)\rvert - \lvert \mathrm{Q}^+(2l-1,q)\rvert$
	(1.3)	$\mathrm{Q}^+(2l-1,q)$	$2\lvert P\mathrm{Q}^+(2l-1,q)\rvert - \lvert \mathrm{Q}^+(2l-1,q)\rvert$
(2)	(2.1)	$L\mathrm{Q}^+(2l-3,q)$	$2\lvert P\mathrm{Q}^+(2l-1,q)\rvert - \lvert L\mathrm{Q}^+(2l-3,q)\rvert$
(3)	(3.1)	$P\mathrm{Q}(2l-2,q)$	$2\lvert \mathrm{Q}(2l,q)\rvert - \lvert P\mathrm{Q}(2l-2,q)\rvert$
	(3.2)	$P\mathrm{Q}(2l-2,q)$	$\lvert \mathrm{Q}(2l,q)\rvert + \lvert P\mathrm{Q}^+(2l-1,q)\rvert - \lvert P\mathrm{Q}(2l-2,q)\rvert$
(4)	(4.1)	$\mathrm{Q}^-(2l-1,q)$	$2\lvert \mathrm{Q}(2l,q)\rvert - \lvert \mathrm{Q}^-(2l-1,q)\rvert$

Table 2

		$\lvert \mathrm{Q}^+(2l+1,q) \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^l + q^{l-2} + \cdots + q + 1$
	(1.2)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 2q^l + q^{l-2} + \cdots + q + 1$
	(1.3)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 3q^l + q^{l-2} + \cdots + q + 1$
(2)	(2.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 2q^l + q^{l-1} + \cdots + q + 1$
(3)	(3.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^l + q^{l-1} + \cdots + q + 1$
	(3.2)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 2q^l + q^{l-1} + \cdots + q + 1$
	(4.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^l + 2q^{l-1} + q^{l-2} + \cdots + q + 1$

Table 3

	Weight	Number of codewords for $q \geqslant 4$
(1.3)	$w_1 = q^{2l} - q^{2l-1} - q^l + q^{l-1}$	$\frac{(q^{3l}+q^{2l})(q^{l+1}-1)}{2}$
(2.1) + (3.2)	$w_1 + q^l - q^{l-1}$	$\frac{(q^{2l+1}-q)(q^{l+1}-1)(q^{l-1}+1)}{2(q-1)} + (q^{3l-1} - q^{l-1})(q^{l+2} - q)$
(1.2)	$w_1 + q^l$	$(q^{3l} + q^{2l})(q^{l+1} - 1)(q - 1)$
(4.1)	$w_1 + 2q^l - 2q^{l-1}$	$\frac{q^{2l+1}(q^{l+1}-1)(q^l-1)(q-1)}{4}$
(3.1)	$w_1 + 2q^l - q^{l-1}$	$\frac{(q^{3l-1}-q^{l-1})(q^{l+1}-1)(q^2-q)}{2}$
(1.1)	$w_1 + 2q^l$	$\frac{(q^{3l}+q^{2l})(q^{l+1}-1)(q^2-3q+2)}{4}$

In the following tables, $\mathrm{Q}^+(2l-1,q)$ and $\mathrm{Q}^-(2l-1,q)$ denote non-singular hyperbolic and elliptic quadrics in $\mathrm{PG}(2l-1,q)$, $P\mathrm{Q}(2l-2,q)$ denotes a singular quadric with vertex the point P and base a non-singular parabolic quadric in $\mathrm{PG}(2l-2,q)$, $L\mathrm{Q}^+(2l-3,q)$ denotes a singular quadric with vertex the line L and base a non-singular hyperbolic quadric in $\mathrm{PG}(2l-3,q)$, $\mathrm{Q}(2l,q)$ denotes a non-singular parabolic quadric in $\mathrm{PG}(2l,q)$, and $P\mathrm{Q}^+(2l-1,q)$ denotes a singular quadric with vertex the point P and base a non-singular hyperbolic quadric in $\mathrm{PG}(2l-1,q)$.

In Table 1, we denote the different possibilities for the intersection of $\mathrm{Q}^+(2l+1,q)$ with the union of two hyperplanes. We describe these possibilities by giving the formula for calculating the size of the intersection. We mention the sizes of the two quadrics which are the intersection of Π_1 and Π_2 with $\mathrm{Q}^+(2l+1,q)$, and we subtract the size of the quadric which is the intersection of $\Pi_{2l-1} = \Pi_1 \cap \Pi_2$ with $\mathrm{Q}^+(2l+1,q)$.

We give in the second table the sizes of these intersections of $\mathrm{Q}^+(2l+1,q)$ with the union of two hyperplanes.

We now present in the third table the weights of the corresponding codewords of $C_2(\mathrm{Q}^+(2l+1,q))$, and the numbers of codewords having these weights.

Remark 4.1. In the case where $q = 2$, we have that the third weight coincides with the fourth. So in that special case there are only five different weights.

4.2. *The elliptic quadric in $PG(2l+1,q)$*

We know that the largest intersections of a non-singular elliptic quadric $\mathrm{Q}^-(2l+1,q)$ in $\mathrm{PG}(2l+1,q)$ with the other quadrics are the intersections of $\mathrm{Q}^-(2l+1,q)$ with the quadrics which are the union of two hyperplanes Π_1 and Π_2. We now discuss all the different possibilities for the intersections of $\mathrm{Q}^-(2l+1,q)$ with the union of two hyperplanes. This then gives the five or six smallest weights of the functional codes $C_2(\mathrm{Q}^-(2l+1,q))$, and the numbers of codewords having these weights.

We again start the discussion via the $(2l-1)$-dimensional space $\Pi_{2l-1} = \Pi_1 \cap \Pi_2$. The intersection of a $(2l-1)$-dimensional space with the non-singular elliptic quadric $\mathrm{Q}^-(2l+1,q)$ in $\mathrm{PG}(2l+1,q)$ is: (1) a non-singular elliptic quadric $\mathrm{Q}^-(2l-1,q)$, (2) a cone $P\mathrm{Q}(2l-2,q)$, (3) a cone $L\mathrm{Q}^-(2l-3,q)$, or (4) a non-singular hyperbolic quadric $\mathrm{Q}^+(2l-1,q)$.

1. Let $\mathrm{PG}(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $\mathrm{Q}^-(2l-1,q)$ in a non-singular $(2l-1)$-dimensional elliptic quadric $\mathrm{Q}^-(2l-1,q)$. Then $\mathrm{PG}(2l-1,q)$ is the polar space of a bisecant line of $\mathrm{Q}^-(2l+1,q)$. Then $\mathrm{PG}(2l-1,q)$ lies in two hyperplanes tangent to $\mathrm{Q}^-(2l+1,q)$ and in $q-1$ hyperplanes intersecting $\mathrm{Q}^-(2l+1,q)$ in a non-singular parabolic quadric $\mathrm{Q}(2l,q)$.

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Table 4

		$\Pi_{2l-1} \cap \mathrm{Q}^-(2l+1,q)$	$\lvert \mathrm{Q}^-(2l+1,q) \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$\mathrm{Q}^-(2l-1,q)$	$2\lvert \mathrm{Q}(2l,q)\rvert - \lvert \mathrm{Q}^-(2l-1,q)\rvert$
	(1.2)	$\mathrm{Q}^-(2l-1,q)$	$\lvert P\mathrm{Q}^-(2l-1,q)\rvert + \lvert \mathrm{Q}(2l,q)\rvert - \lvert \mathrm{Q}^-(2l-1,q)\rvert$
	(1.3)	$\mathrm{Q}^-(2l-1,q)$	$2\lvert P\mathrm{Q}^-(2l-1,q)\rvert - \lvert \mathrm{Q}^-(2l-1,q)\rvert$
(2)	(2.1)	$P\mathrm{Q}(2l-2,q)$	$2\lvert \mathrm{Q}(2l,q)\rvert - \lvert P\mathrm{Q}(2l-2,q)\rvert$
	(2.2)	$P\mathrm{Q}(2l-2,q)$	$\lvert \mathrm{Q}(2l,q)\rvert + \lvert P\mathrm{Q}^-(2l-1,q)\rvert - \lvert P\mathrm{Q}(2l-2,q)\rvert$
(3)	(3.1)	$L\mathrm{Q}^-(2l-3,q)$	$2\lvert P\mathrm{Q}^-(2l-1,q)\rvert - \lvert L\mathrm{Q}^-(2l-3,q)\rvert$
(4)	(4.1)	$\mathrm{Q}^+(2l-1,q)$	$2\lvert \mathrm{Q}(2l,q)\rvert - \lvert \mathrm{Q}^+(2l-1,q)\rvert$

Table 5

		$\lvert \mathrm{Q}^-(2l+1,q) \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$2q^{2l-1}+q^{2l-2}+\cdots+q^l+2q^{l-1}+q^{l-2}+\cdots+q+1$
	(1.2)	$2q^{2l-1}+q^{2l-2}+\cdots+q^{l+1}+2q^{l-1}+q^{l-2}+\cdots+q+1$
	(1.3)	$2q^{2l-1}+q^{2l-2}+\cdots+q^{l+1}-q^l+2q^{l-1}+q^{l-2}+\cdots+q+1$
(2)	(2.1)	$2q^{2l-1}+q^{2l-2}+\cdots+q^{l+1}+q^l+q^{l-1}+\cdots+q+1$
	(2.2)	$2q^{2l-1}+q^{2l-2}+\cdots+q^{l+1}+q^{l-1}+\cdots+q+1$
(3)	(3.1)	$2q^{2l-1}+q^{2l-2}+\cdots+q^{l+1}+q^{l-1}+\cdots+q+1$
(4)	(4.1)	$2q^{2l-1}+q^{2l-2}+\cdots+q^l+q^{l-2}+\cdots+q+1$

Table 6

	Weight	Number of codewords for $q \geqslant 4$
(1.1)	$w_1 = q^{2l}-q^{2l-1}-q^l-q^{l-1}$	$\frac{(q^{3l+1}+q^{2l})(q^l-1)(q^2-3q+2)}{4}$
(2.1)	w_1+q^{l-1}	$\frac{(q^{2l+1}+q^l)(q^{2l}-1)(q-1)}{2}$
(4.1)	w_1+2q^{l-1}	$\frac{q^{2l+1}(q^{l+1}+1)(q^l+1)(q-1)}{4}$
(1.2)	w_1+q^l	$(q^{3l+1}+q^{2l})(q^l-1)(q-1)$
(2.2) + (3.1)	$w_1+q^l+q^{l-1}$	$(q^{2l}+q^{l-1})(q^{2l}-1)q+\frac{(q^{l+2}+q)(q^{2l}-1)(q^{l-1}-1)}{2(q-1)}$
(1.3)	w_1+2q^l	$\frac{(q^{3l+1}+q^{2l})(q^l-1)}{2}$

2. Let $\mathrm{PG}(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $\mathrm{Q}^-(2l+1,q)$ in a singular quadric $P\mathrm{Q}(2l-2,q)$; then $\mathrm{PG}(2l-1,q)$ lies in the hyperplane tangent to $\mathrm{Q}^-(2l+1,q)$ at the point P, and in q hyperplanes intersecting $\mathrm{Q}^-(2l+1,q)$ in non-singular parabolic quadrics $\mathrm{Q}(2l,q)$.
3. Let $\mathrm{PG}(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $\mathrm{Q}^-(2l+1,q)$ in a singular quadric $L\mathrm{Q}^-(2l-3,q)$; then $\mathrm{PG}(2l-1,q)$ lies in the hyperplane tangent to $\mathrm{Q}^-(2l+1,q)$ at the $q+1$ points P of L.
4. Let $\mathrm{PG}(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $\mathrm{Q}^-(2l+1,q)$ in a non-singular $(2l-1)$-dimensional hyperbolic quadric $\mathrm{Q}^+(2l-1,q)$; then $\mathrm{PG}(2l-1,q)$ lies in $q+1$ hyperplanes intersecting $\mathrm{Q}^-(2l+1,q)$ in non-singular parabolic quadrics $\mathrm{Q}(2l,q)$.

In the following tables, $\mathrm{Q}^+(2l-1,q)$ and $\mathrm{Q}^-(2l-1,q)$ denote non-singular hyperbolic and elliptic quadrics in $\mathrm{PG}(2l-1,q)$, $P\mathrm{Q}(2l-2,q)$ denotes a singular quadric with vertex the point P and base a non-singular parabolic quadric in $\mathrm{PG}(2l-2,q)$, $L\mathrm{Q}^-(2l-3,q)$ denotes a singular quadric with vertex the line L and base a non-singular elliptic quadric in $\mathrm{PG}(2l-3,q)$, $\mathrm{Q}(2l,q)$ denotes a non-singular parabolic quadric in $\mathrm{PG}(2l,q)$, and $P\mathrm{Q}^-(2l-1,q)$ denotes a singular quadric with vertex the point P and base a non-singular elliptic quadric in $\mathrm{PG}(2l-1,q)$.

In Table 4, we denote the different possibilities for the intersection of $\mathrm{Q}^-(2l+1,q)$ with the union of two hyperplanes.

We give in the fifth table the sizes of these intersections of $\mathrm{Q}^-(2l+1,q)$ with the union of two hyperplanes.

We present in the sixth table the weights of the corresponding codewords of $C_2(\mathrm{Q}^-(2l+1,q))$, and the numbers of codewords having these weights.

Remark 4.2. In the case where $q=2$, we have that the third weight coincides with the fourth. So in that special case there are only five different weights.

Theorem 4.3. *Let $\mathcal{X}$ be a non-degenerate quadric (hyperbolic or elliptic) in $\mathrm{PG}(2l+1,q)$ where $l \geq 1$. All the weights w_i of the code $C_2(\mathcal{X})$ defined on $\mathcal{X}$ are divisible by q^{l-1}.*

Proof. Let F and f be two forms of degree 2 in $2l+2$ indeterminates with $l \geq 1$ and N the number of common zeros of F and f in $\mathbb{F}_q^{2l+2}$. By the theorem of Ax and Katz [10, p. 85], N is divisible by q^{l-1} since $\frac{2l+2-(2+2)}{2} = l-1$.
On the other hand, F and f are homogeneous polynomials; therefore $N-1$ is divisible by $q-1$. Let $\mathcal{X}$ and $\mathcal{Q}$ be the projective quadrics associated with F and f; one has $|\mathcal{X} \cap \mathcal{Q}| = \frac{N-1}{q-1}$. Let $M = \frac{N-1}{q-1}$; one has

$$M = \frac{kq^{l-1}-1}{q-1} = k\frac{q^{l-1}-1}{q-1} + \frac{k-1}{q-1} = k'q^{l-1} + \pi_{l-2} \qquad (1)$$

where $k, k' \in \mathbb{Z}$ and $k = k'(q-1)+1$. By the theorem of Ax and Katz [10, p. 85] again, we get that the number of zeros of the polynomial F in $\mathbb{F}_q^{2l+2}$ is divisible by q^l, and so

$$|\mathcal{X}| = \frac{tq^l - 1}{q-1} = t\frac{q^l - 1}{q-1} + \frac{t-1}{q-1} = t'q^l + \pi_{l-1} \tag{2}$$

where $t, t' \in \mathbb{Z}$ and $t = t'(q-1)+1$. The weight of a codeword associated with the quadric $\mathcal{X}$ is equal to

$$w = |\mathcal{X}| - |\mathcal{X} \cap \mathcal{Q}| = |\mathcal{X}| - M \tag{3}$$

Therefore, from (1)–(3), we deduce that $w = t'q^l - k'q^{l-1} + q^{l-1}$. □

4.3. The parabolic quadric in PG(2l, q)

We know that the largest intersections of a non-singular parabolic quadric $\mathrm{Q}(2l, q)$ in $\mathrm{PG}(2l, q)$ with the other quadrics are the intersections of $\mathrm{Q}(2l, q)$ with the quadrics which are the union of two hyperplanes Π_1 and Π_2. We now discuss all the different possibilities for the intersections of $\mathrm{Q}(2l, q)$ with the union of two hyperplanes. This then gives the five smallest weights of the functional codes $C_2(\mathrm{Q}(2l, q))$, and the numbers of these codewords.

We proceed as follows. We start the discussion via the $(2l-2)$-dimensional space Π_{2l-2} which is the intersection of the two hyperplanes Π_1 and Π_2. The intersection of a $(2l-2)$-dimensional space with the non-singular parabolic quadric $\mathrm{Q}(2l, q)$ in $\mathrm{PG}(2l, q)$ is: (1) a non-singular parabolic quadric $\mathrm{Q}(2l-2, q)$, (2) a cone $P\mathrm{Q}^+(2l-3, q)$, (3) a cone $P\mathrm{Q}^-(2l-3, q)$, or (4) a cone $L\mathrm{Q}(2l-4, q)$.

For q odd, we can proceed with the discussion via the orthogonal polarity corresponding to the non-singular parabolic quadric $\mathrm{Q}(2l, q)$. For q even, we need to use another approach, since then $\mathrm{Q}(2l, q)$ has a nucleus N. This implies that we need to make a distinction between the $(2l-2)$-dimensional spaces Π_{2l-2} intersecting $\mathrm{Q}(2l, q)$ in a parabolic quadric $\mathrm{Q}(2l-2, q)$ or a quadric $L\mathrm{Q}(2l-4, q)$, containing the nucleus N of $\mathrm{Q}(2l, q)$, and those not containing the nucleus N of $\mathrm{Q}(2l, q)$. In [9, p. 43], these $(2l-2)$-dimensional spaces are respectively called *nuclear* and *non-nuclear*.

We first discuss the case of q odd.

1. Let $\mathrm{PG}(2l-2, q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l, q)$ in a non-singular $(2l-2)$-dimensional parabolic quadric $\mathrm{Q}(2l-2, q)$. Then $\mathrm{PG}(2l-2, q)$ is the polar space of a bisecant or external line of $\mathrm{Q}(2l, q)$. In the first case, $\mathrm{PG}(2l-2, q)$ lies in two hyperplanes tangent to $\mathrm{Q}(2l, q)$, in $(q-1)/2$ hyperplanes intersecting $\mathrm{Q}(2l, q)$ in a non-singular hyperbolic quadric $\mathrm{Q}^+(2l-1, q)$, and in $(q-1)/2$ hyperplanes intersecting $\mathrm{Q}(2l, q)$ in a non-singular elliptic quadric $\mathrm{Q}^-(2l-1, q)$. In the second case, $\mathrm{PG}(2l-2, q)$ lies in $(q+1)/2$ hyperplanes intersecting $\mathrm{Q}(2l, q)$ in a non-singular hyperbolic quadric $\mathrm{Q}^+(2l-1, q)$, and in $(q+1)/2$ hyperplanes intersecting $\mathrm{Q}(2l, q)$ in a non-singular elliptic quadric $\mathrm{Q}^-(2l-1, q)$.
2. Let $\mathrm{PG}(2l-2, q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l, q)$ in a singular quadric $P\mathrm{Q}^+(2l-3, q)$; then $\mathrm{PG}(2l-2, q)$ lies in the hyperplane tangent to $\mathrm{Q}(2l, q)$ in P and in q hyperplanes intersecting $\mathrm{Q}(2l, q)$ in non-singular hyperbolic quadrics $\mathrm{Q}^+(2l-1, q)$.
3. Let $\mathrm{PG}(2l-2, q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l, q)$ in a singular quadric $P\mathrm{Q}^-(2l-3, q)$; then $\mathrm{PG}(2l-2, q)$ lies in the hyperplane tangent to $\mathrm{Q}(2l, q)$ in P, and in q hyperplanes intersecting $\mathrm{Q}(2l, q)$ in non-singular elliptic quadrics $\mathrm{Q}^-(2l-1, q)$.
4. Let $\mathrm{PG}(2l-2, q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l, q)$ in a singular quadric $L\mathrm{Q}(2l-4, q)$; then $\mathrm{PG}(2l-2, q)$ lies in the hyperplanes tangent to $\mathrm{Q}(2l, q)$ in the $q+1$ points P of L.

In the following tables, $\mathrm{Q}(2l-2, q)$ and $\mathrm{Q}(2l, q)$ denote non-singular parabolic quadrics in $\mathrm{PG}(2l-2, q)$ and in $\mathrm{PG}(2l, q)$, $\mathrm{Q}^+(2l-1, q)$ denotes a non-singular hyperbolic quadric in $\mathrm{PG}(2l-1, q)$, $\mathrm{Q}^-(2l-1, q)$ denotes a non-singular elliptic quadric in $\mathrm{PG}(2l-1, q)$, $P\mathrm{Q}(2l-2, q)$ denotes a singular quadric with vertex the point P and base a non-singular parabolic quadric in $\mathrm{PG}(2l-2, q)$, $P\mathrm{Q}^+(2l-3, q)$ denotes a singular quadric with vertex the point P and base a non-singular hyperbolic quadric in $\mathrm{PG}(2l-3, q)$, $P\mathrm{Q}^-(2l-3, q)$ denotes a singular quadric with vertex the point P and base a non-singular elliptic quadric in $\mathrm{PG}(2l-3, q)$, and $L\mathrm{Q}(2l-4, q)$ denotes a singular quadric with vertex the line L and base a non-singular parabolic quadric in $\mathrm{PG}(2l-4, q)$.

In Table 7, we denote the different possibilities for the intersection of $\mathrm{Q}(2l, q)$ with the union of two hyperplanes.

We give in the eighth table the sizes of these intersection of $\mathrm{Q}(2l, q)$ with the union of two hyperplanes.

We now discuss the case of q even. Here $\mathrm{Q}(2l, q)$ has a nucleus N.

1. Let $\mathrm{PG}(2l-2, q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l, q)$ in a non-singular $(2l-2)$-dimensional parabolic quadric $\mathrm{Q}(2l-2, q)$. If $\mathrm{PG}(2l-2, q)$ is non-nuclear, then $\mathrm{PG}(2l-2, q)$ lies in one tangent hyperplane, in the hyperplane $\langle \mathrm{PG}(2l-2, q), N\rangle$, in $q/2$ hyperplanes intersecting $\mathrm{Q}(2l, q)$ in a non-singular hyperbolic quadric $\mathrm{Q}^+(2l-1, q)$, and in $q/2$ hyperplanes intersecting $\mathrm{Q}(2l, q)$ in a non-singular elliptic quadric $\mathrm{Q}^-(2l-1, q)$. If $\mathrm{PG}(2l-2, q)$ is nuclear, then $\mathrm{PG}(2l-2, q)$ lies in $q+1$ hyperplanes tangent to $\mathrm{Q}(2l, q)$.
2. Let $\mathrm{PG}(2l-2, q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l, q)$ in a singular quadric $P\mathrm{Q}^+(2l-3, q)$; then $\mathrm{PG}(2l-2, q)$ lies in the hyperplane tangent to $\mathrm{Q}(2l, q)$ in P, and in q hyperplanes intersecting $\mathrm{Q}(2l, q)$ in non-singular hyperbolic quadrics $\mathrm{Q}^+(2l-1, q)$.

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Table 7

		$\Pi_{2l-2} \cap \mathrm{Q}(2l, q)$	$\lvert \mathrm{Q}(2l, q) \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$\mathrm{Q}(2l-2, q)$	$2\lvert \mathrm{Q}^+(2l-1, q)\rvert - \lvert \mathrm{Q}(2l-2, q)\rvert$
	(1.2)	$\mathrm{Q}(2l-2, q)$	$\lvert \mathrm{Q}^+(2l-1, q)\rvert + \lvert \mathrm{Q}^-(2l-1, q)\rvert - \lvert \mathrm{Q}(2l-2, q)\rvert$
	(1.3)	$\mathrm{Q}(2l-2, q)$	$\lvert P\mathrm{Q}(2l-2, q)\rvert + \lvert \mathrm{Q}^+(2l-1, q)\rvert - \lvert \mathrm{Q}(2l-2, q)\rvert$
	(1.4)	$\mathrm{Q}(2l-2, q)$	$\lvert P\mathrm{Q}(2l-2, q)\rvert + \lvert \mathrm{Q}^-(2l-1, q)\rvert - \lvert \mathrm{Q}(2l-2, q)\rvert$
	(1.5)	$\mathrm{Q}(2l-2, q)$	$2\lvert \mathrm{Q}^-(2l-1, q)\rvert - \lvert \mathrm{Q}(2l-2, q)\rvert$
	(1.6)	$\mathrm{Q}(2l-2, q)$	$2\lvert P\mathrm{Q}(2l-2, q)\rvert - \lvert \mathrm{Q}(2l-2, q)\rvert$
(2)	(2.1)	$P\mathrm{Q}^+(2l-3, q)$	$2\lvert \mathrm{Q}^+(2l-1, q)\rvert - \lvert P\mathrm{Q}^+(2l-3, q)\rvert$
	(2.2)	$P\mathrm{Q}^+(2l-3, q)$	$\lvert \mathrm{Q}^+(2l-1, q)\rvert + \lvert P\mathrm{Q}(2l-2, q)\rvert - \lvert P\mathrm{Q}^+(2l-3, q)\rvert$
(3)	(3.1)	$P\mathrm{Q}^-(2l-3, q)$	$2\lvert \mathrm{Q}^-(2l-1, q)\rvert - \lvert P\mathrm{Q}^-(2l-3, q)\rvert$
	(3.2)	$P\mathrm{Q}^-(2l-3, q)$	$\lvert \mathrm{Q}^-(2l-1, q)\rvert + \lvert P\mathrm{Q}(2l-2, q)\rvert - \lvert P\mathrm{Q}^-(2l-3, q)\rvert$
(4)	(4.1)	$L\mathrm{Q}(2l-4, q)$	$2\lvert P\mathrm{Q}(2l-2, q)\rvert - \lvert L\mathrm{Q}(2l-4, q)\rvert$

Table 8

		$\lvert \mathrm{Q}(2l, q) \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + 3q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.2)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.3)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + 2q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.4)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-2} + \cdots + q + 1$
	(1.5)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l - q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.6)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$
(2)	(2.1)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + 2q^{l-1} + q^{l-2} + \cdots + q + 1$
	(2.2)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$
(3)	(3.1)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-2} + \cdots + q + 1$
	(3.2)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$
(4)	(4.1)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$

Table 9
Weights and number of codewords for q odd.

	Weight	Number of codewords for $q \geqslant 4$
(1.1)	$w_1 = q^{2l-1} - q^{2l-2} - 2q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)(q-3)}{16}$
		$+\frac{q^{2l-1}(q^{2l}-1)(q-1)^2}{16}$
$(1.3) + (2.1)$	$w_1 + q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)}{2} +$
		$\frac{q^l(q^{l-1}+1)(q^{2l}-1)(q-1)}{4}$
(1.2)	$w_1 + 2q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)^2}{8} +$
		$+\frac{q^{2l-1}(q^{2l}-1)(q^2-1)}{8}$
$+(1.6) + (2.2)$		$+\frac{(q^{2l}-1)q^{2l-1}}{2} + \frac{q^l(q^{l-1}+1)(q^{2l}-1)}{2}$
$+(3.2) + (4.1)$		$\frac{q^l(q^{l-1}-1)(q^{2l}-1)}{2} + \frac{(q^{2l}-1)(q^{2l-2}-1)q}{2(q-1)}$
$(1.4) + (3.1)$	$w_1 + 3q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)}{2} + \frac{q^l(q^{l-1}-1)(q^{2l}-1)(q-1)}{4}$
(1.5)	$w_1 + 4q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)(q-3)}{16} + \frac{q^{2l-1}(q^{2l}-1)(q-1)^2}{16}$

3. Let $\mathrm{PG}(2l-2, q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l, q)$ in a singular quadric $P\mathrm{Q}^-(2l-3, q)$; then $\mathrm{PG}(2l-2, q)$ lies in the hyperplane tangent to $\mathrm{Q}(2l, q)$ in P, and in q hyperplanes intersecting $\mathrm{Q}(2l, q)$ in non-singular elliptic quadrics $\mathrm{Q}^-(2l-1, q)$.
4. Let $\mathrm{PG}(2l-2, q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l, q)$ in a singular quadric $L\mathrm{Q}(2l-4, q)$; then $\mathrm{PG}(2l-2, q)$ lies in the hyperplanes tangent to $\mathrm{Q}(2l, q)$ at the $q+1$ points P of L.

In Table 7, we denote the different possibilities for the intersection of $\mathrm{Q}(2l, q)$ with the union of two hyperplanes, and in Table 8, the corresponding sizes for the intersections. We now present in Table 10 the number of codewords having the corresponding weights.

Theorem 4.4. *Let $\mathcal{X}$ be a non-degenerate parabolic quadric in $\mathrm{PG}(2l, q)$ where $l \geq 1$. All the weights w_i of the code $C_2(\mathcal{X})$ defined on $\mathcal{X}$ are divisible by q^{l-1}.*

Proof. It is analogous to that of Theorem 4.3. □

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Table 10
Weights and number of codewords for q even.

	Weight	Number of codewords for $q \geqslant 4$
(1.1)	$w_1 = q^{2l-1} - q^{2l-2} - 2q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-2)(q-1)}{8}$
$(1.3) + (2.1)$	$w_1 + q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)}{2} + \frac{q^l(q^{l-1}+1)(q^{2l}-1)(q-1)}{4}$
$(1.2) + (1.6)$	$w_1 + 2q^{l-1}$	$\frac{(q^{2l}-1)q^{2l}(q-1)}{4} + \frac{q^{2l-1}(q^{2l}-1)}{2} +$
$+(4.1)$		$\frac{q(q^{2l-2}-1)(q^{2l}-1)}{2(q-1)} +$
$+(2.2) + (3.2)$		$\frac{q^l(q^{l-1}+1)(q^{2l}-1)}{2} + \frac{q^l(q^{l-1}-1)(q^{2l}-1)}{2}$
$(1.4) + (3.1)$	$w_1 + 3q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)}{2} + \frac{q^l(q^{l-1}-1)(q^{2l}-1)(q-1)}{4}$
(1.5)	$w_1 + 4q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)(q-2)}{8}$

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References

[1] A. Cossidente, L. Storme, Caps on elliptic quadrics, Finite Fields Appl. 1 (1995) 412–420.
[2] S.V. Droms, K.E. Mellinger, C. Meyer, LDPC codes generated by conics in the classical projective plane, Des. Codes Cryptogr. 40 (2006) 343–356.
[3] F.A.B. Edoukou, Codes correcteurs d'erreurs construits à partir des variétés algébriques, Ph.D. Thesis, Université de la Méditerranée (Aix-Marseille II), France, 2007.
[4] F.A.B. Edoukou, Codes defined by forms of degree 2 on quadric surfaces, IEEE Trans. Inform. Theory 54 (2008) 860–864.
[5] F.A.B. Edoukou, A. Hallez, F. Rodier, L. Storme, On the small weight codewords of the functional codes $C_{herm}(X)$, X a non-singular Hermitian variety, Des. Codes Cryptogr. (submitted for publication).
[6] F.A.B. Edoukou, S. Ling, C. Xing, New information on the structure of the functional codes defined by forms of degree h on non-degenerate Hermitian varieties in $\mathbb{P}^n(\mathbb{F}_q)$, J. Combin. Theory, Ser. A (submitted for publication).
[7] V. Fack, Sz.L. Fancsali, L. Storme, G. Van de Voorde, J. Winne, Small weight codewords in the codes arising from Desarguesian projective planes, Des. Codes Cryptogr. 46 (2008) 25–43.
[8] A. Hallez, L. Storme, Functional codes arising from quadric intersections with Hermitian varieties, Finite Fields Appl. (submitted for publication).
[9] J.W.P. Hirschfeld, J.A. Thas, General Galois Geometries, in: Oxford Mathematical Monographs, Oxford University Press, 1991.
[10] N.M. Katz, On a Theorem of Ax, Amer. J. Math. 93 (1971) 485–499.
[11] J.-L. Kim, K.E. Mellinger, L. Storme, Small weight codewords in LDPC codes defined by (dual) classical generalized quadrangles, Des. Codes Cryptogr. 42 (2007) 73–92.
[12] G. Lachaud, Number of points of plane sections and linear codes defined on algebraic varieties, in: Arithmetic, Geometry, and Coding Theory, Luminy, France, 1993, Walter De Gruyter, Berlin, New York, 1996, pp. 77–104.
[13] M. Lavrauw, L. Storme, G. Van de Voorde, On the code generated by the incidence matrix of points and hyperplanes in $PG(n, q)$ and its dual, Des. Codes Cryptogr. 48 (2008) 231–245.
[14] M. Lavrauw, L. Storme, G. Van de Voorde, On the code generated by the incidence matrix of points and k-spaces in $PG(n, q)$ and its dual, Finite Fields Appl. 14 (2008) 1020–1038.
[15] M. Lavrauw, L. Storme, P. Sziklai, G. Van de Voorde, An empty interval in the spectrum of small weight codewords in the code from points and k-spaces of $PG(n, q)$, J. Combin. Theory, Ser. A 116 (2009) 996–1001.
[16] Z. Liu, D.A. Pados, LDPC codes from generalized polygons, IEEE Trans. Inform. Theory 51 (2005) 3890–3898.
[17] V. Pepe, L. Storme, G. Van de Voorde, Small weight codewords in the LDPC codes arising from linear representations of geometries, J. Combin. Des. 17 (2009) 1–24.
[18] V. Pepe, L. Storme, G. Van de Voorde, On codewords in the dual code of classical generalized quadrangles and classical polar spaces, Discrete Math. (in press).
[19] F. Rodier, A. Sboui, Highest numbers of points of hypersurfaces and generalized Reed–Muller codes, Finite Fields Appl. 14 (2008) 816–822.
[20] A. Sboui, Second highest number of points of hypersurfaces in $\mathbb{F}_{q^n}$, Finite Fields Appl. 13 (2007) 444–449.
[21] A. Sboui, Special numbers of rational points on hypersurfaces in the n-dimensional projective space over a finite field, Discrete Math. 309 (16) (2009) 5048–5059. http://iml.univ-mrs.fr/editions/preprint2006/preprint2006.html.
[22] T. Szőnyi, Zs. Weiner, Private communication, 2009.